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AN EXPANDED TABLE OF PROBABILITY VALUES FOR RAO'S SPACING TEST

Gerald S. Russell

Department of Psychology
1227 University of Oregon
Eugene, OR 97403-1227
jrussell@oregon.uoregon.edu

Daniel J. Levitin

Department of Psychology
1227 University of Oregon
Eugene, OR 97403-1227
levitin@darkwing.uoregon.edu

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ABSTRACT

Rao's Spacing Test is a useful and powerful statistic for testing uniformity with circular and periodic data. Previously published tables of critical values are incomplete, giving critical values only out to p=.1 and n = 200. We extend those tables to n=1000 and for p-values from .001 out to 0.9. The extension will enable researchers to better evaluate the confidence with which they are accepting the null hypothesis, and to use the test with larger n.

INTRODUCTION

The last decade has seen a marked renewal of interest in *circular statistics,* statistical methods employed when the underlying metric of observations is circular or periodic. Examples of circular metrics include time series data, psychophysical measurements of color and musical pitch, and directional measurements of the orienting responses of organisms. With the recent publication of a new monograph on the subject (Fisher, 1993), circular statistics might become more well known and widely used in the years to come.

One particularly useful statistic for testing uniformity of observations is Rao's Spacing Test (Batschelet, 1981; Rao, 1976). In many cases, Rao's test is more powerful than the popular alternatives, such as the Rayleigh Test and Kuiper's V test (Levitin, 1994; Rao, 1972). The test is based on the idea that if the underlying distribution is uniform, successive observations should be approximately evenly spaced, about 360°/N apart. Large deviations from this distribution, resulting from unusually large spaces or unusually short spaces between observations, are evidence for directionality. The test statistic U is defined as:

$$U = \frac{1}{2}\sum_{i=1}^{n}|T_i - \lambda| \tag{1}$$

where $\lambda = 360°/N$

$T_i = f_{i+1} - f_i$ For $1 \leq i \leq n-1$

and $T_n = (360° - f_n) + f_1.$ For $i=n$

Because the sum of the positive deviations must equal the sum of the negative deviations, a simpler computational form eliminates absolute values, so that

$$U = \sum_{i=1}^{n}(T_i - \lambda) \tag{2}$$

summed across positive deviations only.

The density function of U is known to be (Rao, 1976):

$$U = f(u) = (n-1)!\sum_{j=1}^{n-1}\binom{n}{j}(u/2\pi)^{n-j-1}\frac{g_j(nu)}{(n-j-1)!n^{j-1}} \tag{3}$$

for $0 < u < 2\pi[1-(1/n)]$

$$where\ g_j(x) = \left[\frac{1}{2\pi(j-1)!}\right]\sum_{k=0}^{\infty}(-1)^k\binom{j}{k}\left[\left(\frac{x}{2\pi}-k\right)^+\right]^{j-1}$$

Critical values of the U statistic for p = .01, .05 and .1 and n = 1 to 200 were compiled into a table by Rao (1976) and later reproduced by Batschelet (1982). The complexity of the density function compels all but the most dedicated researchers to rely on published tables. We hope that with the publication of this table more investigators will be able to use the test, and in a wider range of circumstances.

COMPUTATIONAL METHOD

If we let $a=x/2\pi$, then the inner sum in the expression for $g_j(x)$ may be written as:

$$h_j(a) = \sum_{k=0}^{\infty}(-1)^k\binom{j}{k}\left[(a-k)^+\right]^{j-1} \qquad (4)$$

This expression is similar to the expression for Sterling's numbers of the second kind (Bogart, 1990); and like Sterling's numbers, it may be evaluated using a recurrence relation,

$$(5) \qquad h_j(a) = a * h_{j-1}(a) + (j-a) * h_{j-1}(a-1)$$

For large n, the terms $h_j(a)$ become very large. We used arbitrary-precision floating point numerics (Press, Teukolsky, Vetterling and Flannery, 1992) to represent the values in the series.

A trapezoidal integration was used to evaluate p given the density function U. The integration was repeated at increasing step sizes until stable values for p were obtained. Finally, a cubic spline interpolation routine was used to find the values shown in the tables below. Table I

TABLE I.

n	U (angles) 45	50	55	60	65	70	75	80	85	90	95	100
4	0.961	0.946	0.928	0.907	0.882	0.853	0.819	0.780	0.736	0.687	0.645	0.602
5	0.983	0.974	0.962	0.946	0.926	0.900	0.870	0.837	0.799	0.758	0.714	0.667
6	0.992	0.987	0.979	0.968	0.952	0.933	0.908	0.879	0.844	0.804	0.759	0.710
7	0.996	0.993	0.988	0.981	0.970	0.954	0.934	0.908	0.877	0.839	0.795	0.745
8	0.998	0.997	0.993	0.988	0.980	0.969	0.952	0.930	0.901	0.866	0.824	0.775
9	0.999	0.998	0.996	0.993	0.987	0.978	0.965	0.946	0.921	0.888	0.848	0.801
10		0.999	0.998	0.996	0.992	0.985	0.974	0.958	0.936	0.907	0.869	0.822
11			0.999	0.997	0.994	0.989	0.981	0.968	0.948	0.921	0.886	0.841
12			0.999	0.998	0.996	0.993	0.986	0.975	0.958	0.934	0.900	0.857
13				0.999	0.998	0.995	0.989	0.980	0.966	0.944	0.913	0.872
14				0.999	0.998	0.996	0.992	0.985	0.972	0.952	0.924	0.884
15					0.999	0.997	0.994	0.988	0.977	0.959	0.933	0.895
16					0.999	0.998	0.996	0.990	0.981	0.965	0.941	0.905
17						0.999	0.997	0.992	0.985	0.971	0.948	0.914
18						0.999	0.997	0.994	0.987	0.975	0.954	0.922
19						0.999	0.998	0.995	0.989	0.978	0.960	0.930
20							0.999	0.996	0.991	0.982	0.964	0.936
21							0.999	0.997	0.993	0.984	0.968	0.942
22							0.999	0.998	0.994	0.986	0.972	0.947
23							0.999	0.998	0.995	0.988	0.975	0.952
24								0.999	0.996	0.990	0.978	0.956
25								0.999	0.997	0.991	0.980	0.960
26								0.999	0.997	0.993	0.983	0.963
27								0.999	0.998	0.994	0.985	0.966
28								0.999	0.998	0.995	0.986	0.969
29									0.998	0.995	0.988	0.972
30									0.999	0.996	0.989	0.974
35									0.999	0.998	0.994	0.984
40										0.999	0.997	0.989
45											0.998	0.993
50											0.999	0.995
75												0.999

shows p values for n=4 to n=1000 when the value of the test statistic U is known. Table II the value of the test statistic U when given the parameters n and p, again, for n=4 to n=1000, and for stepped p values between 0.001 and 0.9.

The values we obtained differ slightly in some cases from those obtained by Rao two decades ago. Perhaps this is due to our use of arbitrary precision arithmetic and relatively small step sizes in our calculations, which were made possible by the high speed and large memory capacity of present day computers.

TABLE I. Continued.

n \ U	105	110	115	120	125	130	135	140	145	150	155	160
4	0.559	0.515	0.471	0.428	0.386	0.345	0.305	0.267	0.230	0.197	0.166	0.138
5	0.617	0.566	0.514	0.461	0.410	0.360	0.313	0.270	0.233	0.199	0.169	0.141
6	0.657	0.601	0.544	0.488	0.434	0.381	0.330	0.283	0.239	0.199	0.163	0.132
7	0.691	0.634	0.574	0.514	0.453	0.394	0.338	0.286	0.239	0.196	0.160	0.128
8	0.721	0.661	0.598	0.533	0.469	0.406	0.346	0.290	0.239	0.193	0.154	0.120
9	0.746	0.685	0.619	0.551	0.483	0.415	0.351	0.291	0.237	0.189	0.148	0.114
10	0.768	0.706	0.638	0.567	0.495	0.423	0.355	0.292	0.235	0.185	0.143	0.108
11	0.787	0.724	0.655	0.582	0.506	0.431	0.359	0.292	0.232	0.181	0.137	0.101
12	0.804	0.741	0.671	0.595	0.516	0.437	0.362	0.292	0.230	0.176	0.132	0.096
13	0.819	0.757	0.685	0.607	0.525	0.443	0.364	0.291	0.227	0.172	0.126	0.090
14	0.833	0.771	0.698	0.618	0.533	0.448	0.366	0.290	0.224	0.167	0.121	0.085
15	0.846	0.784	0.710	0.628	0.541	0.453	0.368	0.289	0.221	0.163	0.116	0.080
16	0.857	0.796	0.722	0.638	0.548	0.457	0.369	0.288	0.218	0.158	0.111	0.075
17	0.868	0.807	0.732	0.647	0.555	0.461	0.370	0.287	0.214	0.154	0.107	0.071
18	0.877	0.817	0.742	0.656	0.562	0.465	0.371	0.286	0.211	0.150	0.102	0.067
19	0.886	0.826	0.752	0.664	0.568	0.469	0.372	0.284	0.208	0.146	0.098	0.063
20	0.894	0.835	0.761	0.672	0.574	0.472	0.373	0.283	0.205	0.142	0.094	0.060
21	0.901	0.844	0.769	0.680	0.579	0.475	0.374	0.281	0.202	0.138	0.090	0.056
22	0.908	0.852	0.777	0.687	0.585	0.478	0.374	0.279	0.199	0.135	0.087	0.053
23	0.914	0.859	0.785	0.694	0.590	0.481	0.374	0.278	0.196	0.131	0.083	0.050
24	0.920	0.866	0.792	0.700	0.595	0.484	0.375	0.276	0.193	0.128	0.080	0.047
25	0.925	0.872	0.799	0.707	0.600	0.486	0.375	0.274	0.190	0.124	0.077	0.045
26	0.930	0.878	0.806	0.713	0.604	0.489	0.375	0.273	0.187	0.121	0.074	0.042
27	0.935	0.884	0.812	0.719	0.609	0.491	0.375	0.271	0.184	0.118	0.071	0.040
28	0.939	0.890	0.818	0.724	0.613	0.493	0.376	0.269	0.181	0.115	0.068	0.038
29	0.943	0.895	0.824	0.730	0.617	0.496	0.376	0.268	0.179	0.112	0.065	0.035
30	0.946	0.900	0.829	0.735	0.622	0.498	0.376	0.266	0.176	0.109	0.063	0.034
35	0.961	0.920	0.854	0.759	0.640	0.508	0.375	0.258	0.163	0.095	0.051	0.025
40	0.972	0.936	0.874	0.780	0.657	0.516	0.374	0.249	0.152	0.084	0.042	0.019
45	0.979	0.949	0.891	0.798	0.672	0.523	0.373	0.241	0.141	0.074	0.035	0.015
50	0.985	0.959	0.905	0.815	0.685	0.530	0.372	0.234	0.131	0.065	0.029	0.011
75	0.997	0.985	0.952	0.874	0.740	0.557	0.363	0.200	0.093	0.036	0.011	0.003
100	0.999	0.995	0.974	0.912	0.779	0.577	0.353	0.173	0.067	0.020	0.005	0.001
150		0.999	0.992	0.955	0.836	0.607	0.335	0.132	0.036	0.007	0.001	
200			0.998	0.976	0.875	0.631	0.318	0.102	0.020	0.002		
300				0.993	0.924	0.668	0.289	0.063	0.006			
400				0.998	0.953	0.697	0.265	0.039	0.002			
500				0.999	0.970	0.721	0.245	0.025	0.001			
600					0.980	0.742	0.227	0.016				
700					0.987	0.760	0.210	0.011				
800					0.992	0.776	0.196	0.007				
900					0.994	0.791	0.183	0.005				
1000					0.996	0.804	0.171	0.003				

(continued)

TABLE I. Continued.

n \ U	165	170	175	180	185	190	195	200	205	210	215	220
4	0.113	0.092	0.075	0.063	0.053	0.044	0.036	0.029	0.024	0.019	0.014	0.011
5	0.116	0.094	0.076	0.060	0.046	0.035	0.026	0.020	0.015	0.011	0.008	0.006
6	0.106	0.084	0.066	0.052	0.040	0.030	0.022	0.016	0.011	0.008	0.005	0.004
7	0.100	0.077	0.059	0.044	0.032	0.024	0.017	0.012	0.008	0.006	0.004	0.002
8	0.093	0.070	0.052	0.038	0.027	0.019	0.013	0.009	0.006	0.004	0.002	0.002
9	0.086	0.063	0.046	0.032	0.023	0.015	0.010	0.007	0.004	0.003	0.002	0.001
10	0.079	0.057	0.040	0.028	0.019	0.012	0.008	0.005	0.003	0.002	0.001	0.001
11	0.073	0.052	0.035	0.024	0.016	0.010	0.006	0.004	0.002	0.001	0.001	
12	0.068	0.047	0.031	0.020	0.013	0.008	0.005	0.003	0.002	0.001		
13	0.063	0.042	0.027	0.017	0.011	0.006	0.004	0.002	0.001	0.001		
14	0.058	0.038	0.024	0.015	0.009	0.005	0.003	0.002	0.001			
15	0.053	0.034	0.021	0.013	0.007	0.004	0.002	0.001	0.001			
16	0.049	0.031	0.019	0.011	0.006	0.003	0.002	0.001				
17	0.046	0.028	0.017	0.009	0.005	0.003	0.001	0.001				
18	0.042	0.025	0.015	0.008	0.004	0.002	0.001					
19	0.039	0.023	0.013	0.007	0.004	0.002	0.001					
20	0.036	0.021	0.011	0.006	0.003	0.001	0.001					
21	0.033	0.019	0.010	0.005	0.002	0.001						
22	0.031	0.017	0.009	0.004	0.002	0.001						
23	0.028	0.015	0.008	0.004	0.002	0.001						
24	0.026	0.014	0.007	0.003	0.001	0.001						
25	0.024	0.013	0.006	0.003	0.001							
26	0.023	0.011	0.005	0.002	0.001							
27	0.021	0.010	0.005	0.002	0.001							
28	0.019	0.009	0.004	0.002	0.001							
29	0.018	0.009	0.004	0.002	0.001							
30	0.017	0.008	0.003	0.001								
35	0.011	0.005	0.002	0.001								
40	0.008	0.003	0.001									
45	0.005	0.002	0.001									
50	0.004	0.001										
75	0.001											

EXAMPLES

Both example 1 and example 2 were taken from Levitin (1994). Fig. 1 displays hypothetical data for onset time of seizures in an epileptic patient, collected across several weeks. The data form a continuous circular distribution, and times of day are converted into angles around the circle. 0° = 12 midnight; each hour = 360°/24 = 15°; minutes = 15°/60 = .25°. The experimental question is whether the seizures are uniformly distributed throughout the day, or whether

TABLE II.

n	p .001	.005	.01	.05	.10	.50	.90
4	247.32	231.22	221.14	186.45	168.02	111.72	61.48
5	245.19	223.98	211.93	183.44	168.66	116.30	69.98
6	236.81	216.05	206.79	180.65	166.30	118.95	76.52
7	229.46	211.57	202.55	177.83	165.05	121.13	81.43
8	224.41	206.91	198.46	175.68	163.56	122.57	85.23
9	219.52	203.17	195.27	173.68	162.36	123.74	88.35
10	215.44	199.91	192.37	171.98	161.23	124.64	90.95
11	211.87	197.04	189.88	170.45	160.24	125.38	93.15
12	208.69	194.51	187.66	169.09	159.33	125.99	95.05
13	205.87	192.25	185.68	167.87	158.50	126.50	96.71
14	203.33	190.23	183.90	166.76	157.75	126.94	98.17
15	201.04	188.39	182.28	165.75	157.06	127.32	99.47
16	198.96	186.72	180.81	164.83	156.43	127.64	100.64
17	197.05	185.19	179.46	163.98	155.84	127.93	101.70
18	195.29	183.78	178.22	163.20	155.29	128.19	102.67
19	193.67	182.48	177.08	162.47	154.78	128.42	103.55
20	192.17	181.27	176.01	161.79	154.31	128.62	104.36
21	190.78	180.15	175.02	161.16	153.86	128.81	105.10
22	189.47	179.10	174.10	160.56	153.44	128.98	105.80
23	188.25	178.11	173.23	160.01	153.05	129.13	106.44
24	187.11	177.19	172.41	159.48	152.68	129.27	107.04
25	186.03	176.32	171.64	158.99	152.32	129.40	107.61
26	185.01	175.50	170.92	158.52	151.99	129.52	108.13
27	184.05	174.73	170.23	158.07	151.67	129.63	108.63
28	183.14	173.99	169.58	157.65	151.37	129.73	109.10
29	182.28	173.29	168.96	157.25	151.08	129.82	109.54
30	181.45	172.63	168.38	156.87	150.80	129.91	109.96
35	177.88	169.74	165.81	155.19	149.59	130.28	111.76
40	174.99	167.39	163.73	153.82	148.60	130.55	113.20
45	172.58	165.44	162.00	152.68	147.76	130.76	114.38
50	170.54	163.79	160.53	151.70	147.05	130.93	115.37
75	163.60	158.13	155.49	148.34	144.56	131.44	118.68
100	159.45	154.74	152.46	146.29	143.03	131.69	120.62
150	154.51	150.69	148.84	143.83	141.18	131.94	122.88
200	151.56	148.26	146.67	142.35	140.06	132.06	124.21
300	148.06	145.38	144.09	140.57	138.71	132.19	125.76
400	145.96	143.66	142.54	139.50	137.89	132.25	126.68
500	144.54	142.48	141.48	138.77	137.33	132.29	127.30
600	143.48	141.60	140.70	138.23	136.91	132.31	127.76
700	142.66	140.93	140.09	137.80	136.59	132.33	128.11
800	142.00	140.38	139.60	137.46	136.33	132.34	128.40
900	141.45	139.93	139.19	137.18	136.11	132.35	128.63
1000	140.99	139.54	138.84	136.94	135.92	132.36	128.83

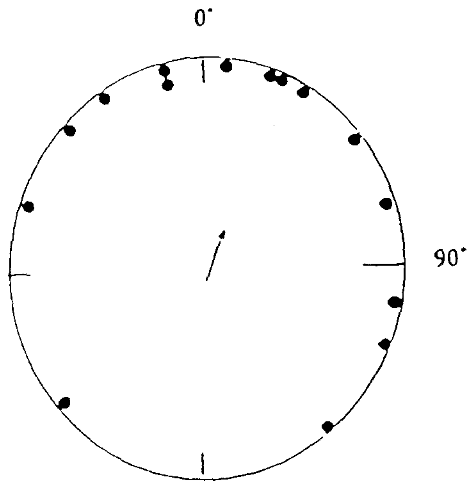

FIG. 1. Circular ordering of time-of-day seizure data (hypothetical) from Example 1.

TABLE III.

arc length = l = 360/N = 24°

i	Time	*f*	T_i	$\|T_i - l\|$
1	12:25 am	6.5°	13.5	10.5
2	1:20 am	20°	2.8	21.2
3	1:31 am	22.8°	7.2	16.8
4	2:00 am	30°	20.1	3.9
5	3:20 am	50°	20.0	4.0
6	4:40 am	70°	28.0	8.0
7	6:32 am	98°	12.0	12.0
8	7:20 am	110°	30.0	6.0
9	9:20 am	140°	90.0	66.0
10	3:20 pm	230°	60.0	36.0
11	7:20 pm	290°	25.0	1.0
12	9:00 pm	315°	15.0	9.0
13	10:00 pm	330°	18.0	6.0
14	11:12 pm	348°	0	24.0
15	11:12 pm	348°	18.5	5.5
N=15			Σ =226	

U (15) = 113; p>.74.

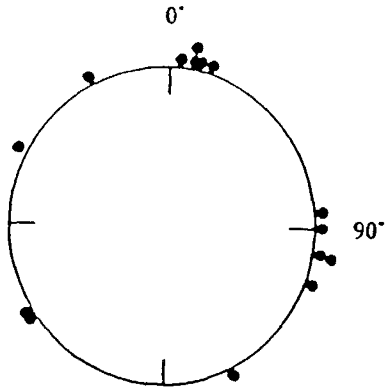

FIG. 2. Circular ordering of time-of-day seizure data (hypothetical) from Example 2.

there are increased seizures at particular times. The times for seizure onsets are: 12:26 am, 1:20 am, 1:31 am, 2:00 am, 3:20 am, 4:40 am, 6:32 am, 7:20 am, 9:20 am, 3:20 pm, 7:20 pm, 9:00 pm, 10:00 pm, 11:12 pm, 11:12 pm; the equivalent angular displacements, *f*, are: 6.5°, 20°, 22.8°, 30°, 50°, 70°, 98°, 110°, 140°, 230°, 290°, 315°, 330°, 348°, 348°. The computations for the Rao test in the epileptic seizure example are shown in Table III. The Rao test yields U=113, p>.74. A Rayleigh test of the same distribution yields p=.033. This result illustrates that the Rao test is not the most powerful in all cases. In particular, it is less powerful than alternative tests when the distribution is unimodal or multi-modal on a global scale, yet simultaneously very uniform over local regions of the distribution. However, in other cases, Rao's test has greater power than alternatives, as shown in the following example.

Fig. 2 displays another set of hypothetical data for a different epileptic patient recording the times for onset of seizures. In this example, the angular values of the times are: 5, 10, 10, 12, 17, 85, 90, 99, 100, 110, 153, 233, 235, 296 and 331. In this case, under Rao's spacing test, we find U=177, p<.02. By comparison, the Rayleigh test yields p=.21. The Rao test is more sensitive to the non-uniformity of this distribution because it is more locally "clumpy" than the first example.

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