PSYCHOMETRIKA—VOL. 73, NO. 2, 321–339
JUNE 2008
DOI: 10.1007/s11336-007-9032-1

FROM ZERO TO SIXTY: CALIBRATING REAL-TIME RESPONSES

THEODORO KOULIS, JAMES O. RAMSAY, AND DANIEL J. LEVITIN

MCGILL UNIVERSITY

Recent advances in data recording technology have given researchers new ways of collecting online and continuous data for analyzing input–output systems. For example, continuous response digital interfaces are increasingly used in psychophysics. The statistical problem related to these input–output systems reduces to linking time-varying covariates to a continuous response variate. Using real-time data obtained from an experiment in psychoacoustics, we showcase new statistical tools that incorporate dynamical elements of an input–output system. We employ functional data analysis (FDA) methods and a simple differential equation to analyze and model the continuous responses. Furthermore, we outline the issues involved in analyzing input–output systems when the exact form of the underlying mathematical model is not known. Finally, we develop a calibration method to facilitate inter-subject and intra-subject comparisons.

Key words: input–output, functional data analysis, real-time data, differential equations, response calibration.

1. Introduction

The collection of real-time data is becoming increasingly popular for exploring the time-varying characteristics of behaviour, and often involves digital interfaces for the recording of continuous multidimensional responses generated by a manipulandum of some kind, such as a slider, dial or mouse. In these experiments, subjects often monitor signals presented in various sensory modalities, and are required to adjust their responses to reflect changes in these time-varying or functional inputs. The modelling of the *change* in these responses as modulated by both functional and multivariate covariates is what we refer to as the study of response *dynamics*. We introduce ideas from functional data analysis, differential equations and mixed-effects modelling with a view to modelling how change in behaviour is related to various multivariate and functional inputs or covariates.

Research on behavioural dynamics dates back to at least World War II, when behavioural scientists studied the corrective movements of aircraft gunners as they tracked targets. Two early papers are Searle and Taylor (1948) and Taylor and Birmingham (1948), and new research and methods are reported in Walls and Schafer (2006) and Chow, Ferrer and Nesselroade (in press). Sliders have been used in music cognition by Madsen and Fredrickson (1993) and by Vines, Krumhansl, Wanderley and Levitin (2006) to record continuous judgements of tension and of phrasing in a musical performance. The concepts and mathematical techniques of process control, used in engineering to design and evaluate feedback procedures to stabilize industrial processes, have been applied to biology (Grodins, 1963), physiology (Wiener, 1948), economics

This work was supported by grants from the National Sciences and Engineering Research Council (NSERC) to J. O. Ramsay and to D. J. Levitin, and by a grant from the Social Sciences and Humanities Research Council (SSHRC) to D. J. Levitin.

We would like to thank Bennett Smith for designing and implementing the software used to conduct the pitch tracking experiment. Also, we wish to thank the research assistants in the Levitin Laboratory involved in the data collection: Catherine Chapados, Andrew Schaaf and Carla Himmelman. We would also like to acknowledge Giles Hooker's work on implementing the generalized profiling software used within this paper.

Requests for reprints should be sent to James O. Ramsay, Department of Psychology, McGill University, Montreal, Quebec, Canada. E-mail: ramsay@psych.mcgill.ca

(Tustin, 1952) and psychology (Boker & Wenger 2007; Jagacinski & Flach 2003a; Marken, 2001; Newell & Molenaar 1998; Powers, 1973, 1990; Runkel, 1990).

We showcase recent developments in functional data analysis to extract meaning from time series data by allowing us to estimate the parameters that govern the relationships between stimuli and dynamic responses. We have noted large individual differences in these responses, and, using mixed or hierarchical nonlinear models, we estimate how to estimate both population parameters and individual performance features, and how to fine-tune the mathematical formulations of the dynamical responses.

2. The Pitch Tracking Experiment

Consider the following experiment. A participant is asked to follow a series of tones of randomly varying frequencies and lengths by adjusting a slider on a computer input device. The input device does not influence the sound of the tones and no feedback is given. It is simply a way to measure the participant's subjective response to the changes in frequency—the pitch distances between tones and the temporal intervals at which they change. Given that the range of the slider represents a finite range of frequencies, the participant is instructed to move the slider in the direction of the change in pitch; if the pitch increases, the participant is expected to move the slider towards a position representative of this new level. The computer input device samples the responses at a frequency of 172.27 Hz, or every 0.0058 seconds, and aligns the data with the corresponding pitch history. Pooling the data from all participants ($n = 16$) gives a rich collection of continuously tracked stimuli and responses with strong serial correlations.

The top panel of Figure 1 displays a step function representing the auditory stimulus as a sequence of constant pitches, and the middle panel contains a plot of a single participant's continuous adjustments. These plots show that the slider position follows the changing pitch levels, but a closer inspection reveals that responses are not always consistent for equal changes in the stimulus. Furthermore, there are differences in the way participants of the experiment use the slider: some participants are prone to using a small range of the slider to map out frequencies, while others use the full range of the slider, often hitting the boundaries of the interface. Plotting the responses of all $n = 16$ participants (bottom panel of Figure 1) reveals large inter-subject variation that may be regarded as individual random effects. For any meaningful conclusions to be made, a proper calibration of individual responses needs to be carried out. Before commencing a detailed discussion of calibration, we begin with a description of a first-order step response system to describe the relationship between response and stimulus in the pitch tracking experiment.

Each participant was presented with three consecutive two-minute streams of stimuli (three blocks; see *Procedure* in Appendix), where each stimulus consisted of a steady-state sine tone generated at random where the length of duration was sampled from a rectangular distribution and the frequency was chosen at random on a log-linear scale in the three-octave interval from D3 (146.8 Hz) to D6 (1174.7 Hz).

Approximately six minutes of stimulus and response histories were recorded for every participant, and these are denoted by $\mathbf{U}_i$ and $\mathbf{y}_i$, respectively. Since humans perceive pitch on an approximately log-linear scale (Stevens & Volkmann, 1940), we let $\mathbf{u}_i = \log_2 \mathbf{U}_i$ be the perceived input function for individual i and associate a vector of observation times $\mathbf{t}_i$ to the recorded histories.

3. Components of Perception–Action Systems

Within the domain of psychophysics, input–output (IO) systems are referred to as perception–action (PA) systems (Jagacinski & Flach, 2003b), and the pitch tracking experiment is

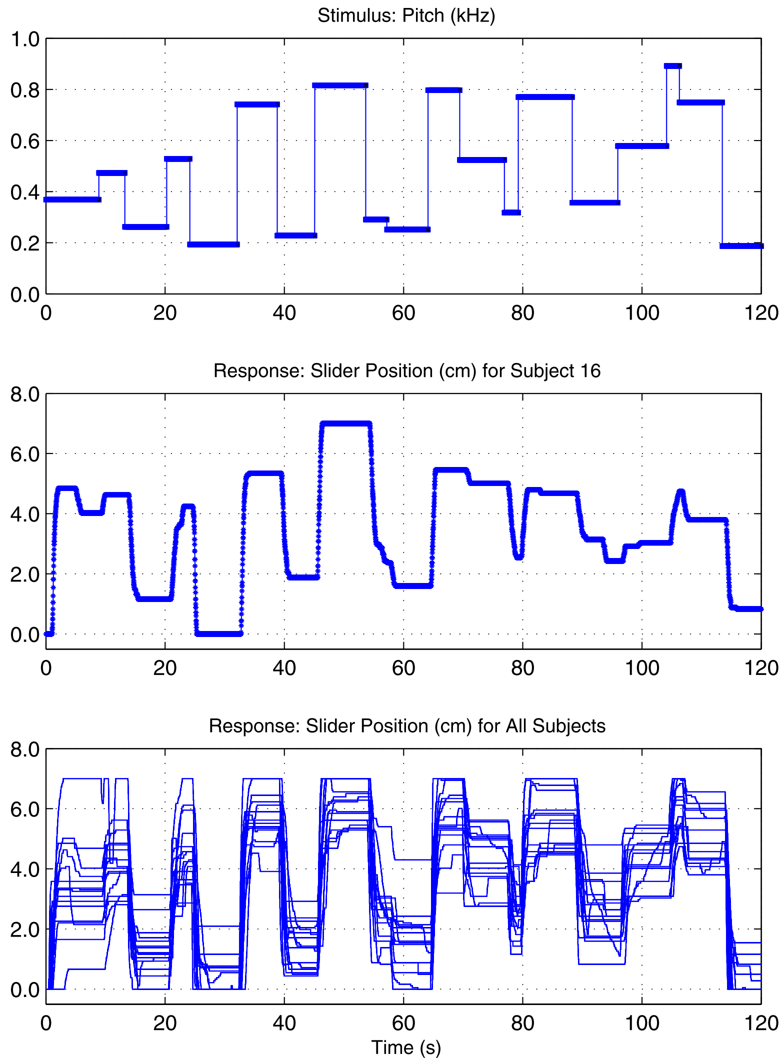


FIGURE 1.
The slider experiment as a PA system. *Top*: A sequence of sine tones at various frequencies. *Middle*: A participant hears these pitches and moves the slider position accordingly. *Bottom*: Responses of all $n = 16$ participants with the same input.

one example. For many real-world scenarios, the organization and behaviour of the components within the system are not easily observable. When viewed as a black box system, the external and observable behaviour of the system is the relationship between the input and output histories. The ensemble of components comprising the PA system is considered to be greater than the sum of its parts, and it is the role of the scientist to determine the law and properties of the system.

Within the context of the pitch tracking experiment, each individual represents a unique black box composed of the participant's motor and cognitive processes, the electronic measurement system consisting of the computer (hardware and software) and the slider device. The interactions between the various cognitive components are not completely understood and we can only observe the pitch function (input) and the slider position (output). Theoretical knowledge

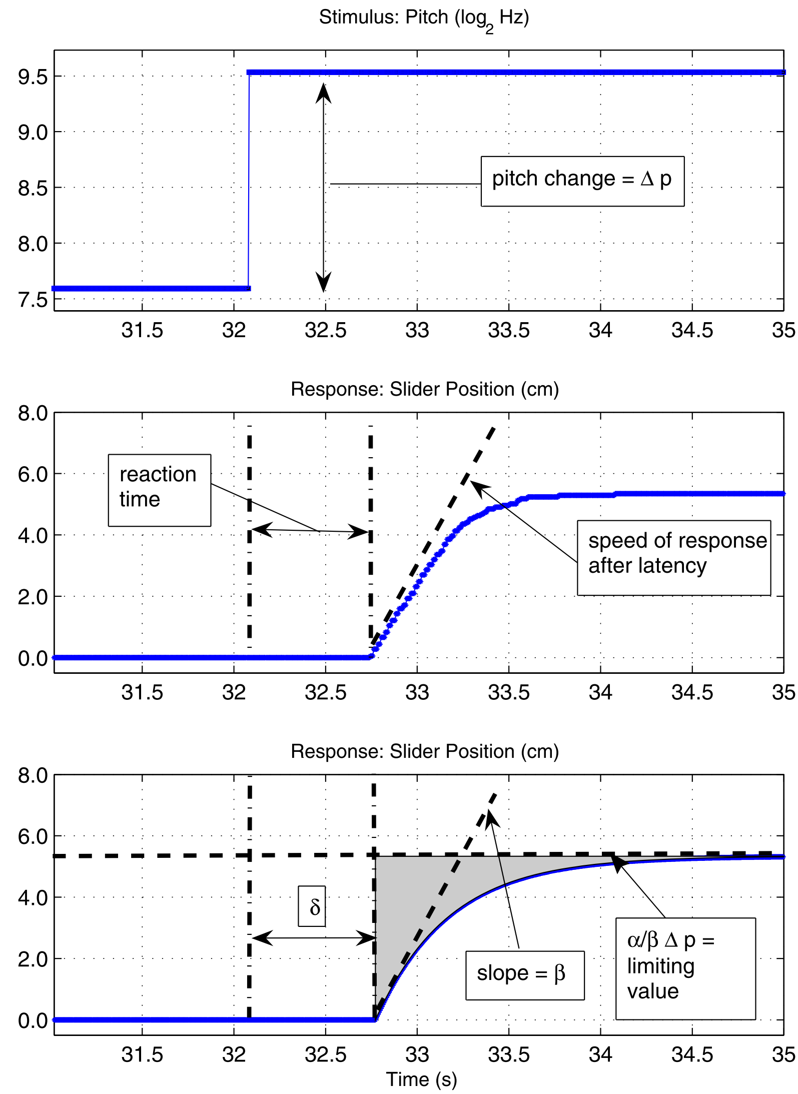


FIGURE 2.
Top: A step function representing a forcing function. *Middle*: A participant's reaction to the stimulus. *Bottom*: A solution to a first-order differential equation with $\alpha = 6.87$, $\beta = 2.5$ and $\delta = 0.73$. Also, $\Delta p = 1.94$, so $x^* = 5.33$. *Shaded*: The reluctance of the subject's response to the stimulus.

about the internal workings of a system can lead to mathematical formulations which are refined with further experimentation.

3.1. Reaction Time, Response Speed and Gain

The three fundamental features of a continuous response are: reaction time, response speed and gain. Reaction time (δ) is the latency period that elapses between the onset of a fixed stimulus and the response to it. In the pitch tracking experiment described above, the participant's reaction time depends on a combination of factors: the ability of an individual to search and detect a signal, to interpret its meaning, to decide on an appropriate reference level, to plan and initiate a motor response and an error term. This error term can be further decomposed into decision noise, neural noise, participant effects and other random effects. Response speed (β), also known as movement speed, is how fast an individual implements the response to the stimulus after the latency period. Gain (G) is the ratio of output change (response) to input change (stimulus).

A portion of the slider data from Figure 1 is displayed in Figure 2. The latency period is delineated by two vertical lines and gain is calculated by taking the final slider position, y^*, and dividing it by the perceived change in pitch, p. The participant's response speed after the latency period is represented by the slope of the tangent to the slider position (centre panel of Figure 2), which suggests that this speed is variable in time. To incorporate these features, we considered a differential equation with three structural parameters that are related to reaction time, response speed and gain.

3.2. First-Order Step Response

Time-varying systems may be described mathematically as dynamical systems. Some examples are equations used to describe the motion of an object subjected to an external force, such as the swinging of a pendulum, the motion of a spring or the orbit of a planet. More complex models may be used to describe the flow of water as it passes through a pipe, the diffusion of heat in a metal, or the motion of a cursor on a computer screen in a tracking experiment.

The time-evolution of a dynamical system with response vector $\mathbf{x}(t)$ is specified by a system of differential equations:

$$\mathrm{D}\mathbf{x}(t) = \mathbf{f}(\mathbf{x}, \mathbf{u}, t|\boldsymbol{\theta}), \tag{1}$$

where $\mathrm{D}\mathbf{x}(t)$ denotes the first derivative of $\mathbf{x}(t)$ and $\mathbf{f}$ is a function vector describing the pattern of change in $\mathbf{x}$ in relation to the stimulus vector $\mathbf{u}$ at time t. The vector $\boldsymbol{\theta}$ is an array of structural parameters that govern the system; once a form for $\mathbf{f}$ is proposed, interest lies in identifying the parameters of the system.

Equation (1) is a first-order differential equation since it only includes the first derivative of the response vector. We will restrict our discussion to first-order dynamics, since systems with higher-order derivatives $\mathrm{D}^m\mathbf{x}(t)$, $m \geq 2$, may be reduced to the form of (1). For a system of order m with one output variable $x(t)$, this is achieved with the vector $\mathbf{x}(t)$ with components $x_1(t) = x(t), x_2(t) = \mathrm{D}x(t), \ldots, x_{m-1}(t) = \mathrm{D}^{m-1}x(t)$. An application of the second derivative in music perception has been previously explored in Vines, Nuzzo and Levitin (2005).

For the pitch tracking experiment, we consider a system with a single continuous response $x(t)$ that represents the slider position and that presumably corresponds to the pitch stimulus $u(t)$. The most basic dynamic system at our disposal to describe the motions of the slider is the first-order step response with a delayed forcing term:

$$\mathrm{D}x(t) = -\beta x(t) + \alpha u(t - \delta), \tag{2}$$

where $\mathrm{D}x(t)$ represents the response velocity. The constant structural parameters α, β and δ are assumed positive and relate to gain, response speed and reaction time.

The parameter δ in (2) represents the reaction time of the system since it acts to shift the forcing function $u(t)$ to the right by δ time units. The roles of α and β are better understood by examining the general solution to the first-order step response system (2). If the initial position of the slider is $x(0) = 0$, the solution of (2) may be expressed as

$$x(t) = \alpha \exp\{-\beta(t - \delta)\} \int_0^{t-\delta} \exp\{\beta s\} u(s)\, \mathrm{d}s. \tag{3}$$

In our psychoacoustics experiment, stimulus $u(t)$ is a step function with magnitude p:

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ p & \text{if } t \geq 0, \end{cases} \tag{4}$$

and the solution of the slider position (3) reduces to

$$x(t) = \begin{cases} 0 & \text{if } t < \delta, \\ (\alpha/\beta)p(1 - \exp\{-\beta(t-\delta)\}) & \text{if } t \geq \delta. \end{cases} \tag{5}$$

Figure 2 displays the plots of a step function $u(t)$ of magnitude $p = 1.94$ (top panel) and the corresponding solution $x(t)$ (bottom panel) where $\alpha = 6.87$, $\beta = 2.5$ and $\delta = 0.73$.

3.3. *Behaviour of the Solution*

From (5) one can deduce that the limiting value x^* of the solution $x(t)$ is $x^* = x(\infty) = (\alpha/\beta)p$. In control theory, the ratio $G = \alpha/\beta$ is called the steady-state gain of the system and is calculated by taking the limiting value x^* and dividing by the change in input p:

$$G = \frac{x^*}{p}. \tag{6}$$

The gain can be viewed as a scaling factor that relates the magnitude of change in input to the response function $x(t)$. It may also be convenient to see the gain as a measure of "effort" or "energy" required to reach the final value of the response. In Figure 2 we have a situation where the pitch change is $p = 1.94$, $\delta = 0.73$, $\beta = 2.5$ and $\alpha = 6.87$. In this case, the gain is $G = 2.75$ or $x^* \approx 5.33$.

The slope of the response $x(t)$ after the latency period is given by $\mathrm{D}x(\delta) = \alpha p$. Were this rate of change to be maintained, the response would reach the limiting value x^* in exactly $1/\beta$ time units, however, this does not occur in the example presented in Figure 2. In fact, the response velocity slows down as $x(t)$ approaches the steady-state position, x^*. After $\delta + 1/\beta$ time units have elapsed, the solution attains the level $x(\delta + 1/\beta) = 0.632x^*$, or approximately two-thirds of the final value x^*. After $\delta + 2/\beta$ time units, the solution $x(t)$ has attained approximately seven-eighths of the final value and after $\delta + 4/\beta$ time units, the solution is practically (98.17%) at the steady state. Because of this, the ratio $\tau = 1/\beta$ is often referred to in engineering textbooks as the system time constant. The parameter β will denote the response speed and should be treated as a measure of how fast an individual approaches the steady state.

For a real-world experiment, we assume that the response speed β is not zero; we can thus rewrite the first-order equation (2) in terms of G, δ and τ:

$$\tau \mathrm{D}x(t) = -x(t) + Gu(t - \delta). \tag{7}$$

When $\tau = 0$, the response speed is infinite and the response curve $x(t)$ reacts instantaneously to the latent stimulus $u(t - \delta)$. This instantaneous response is denoted by $x^0(t)$ and is a multiple of the delayed stimulus:

$$x^0(t) = Gu(t - \delta). \tag{8}$$

An overall measure of the difference between the instantaneous response $x^0(t)$ and the actual response is constructed by taking an integrated difference:

$$R = \int_0^\infty \left[x^0(s) - x(s)\right] \mathrm{d}s = Gp \times \int_\delta^\infty \left[1 - \left(1 - \exp\left\{-\frac{s-\delta}{\tau}\right\}\right)\right] \mathrm{d}s = \tau Gp. \tag{9}$$

R is called the *reluctance* of the response and it is treated as a measure of hesitation in responding to a given stimulus. For the example represented in Figure 2, the reluctance is $R \approx 2.13$ and this represents the area of the shaded region in the bottom panel. Presumably, R is a function of task difficulty and reflects the time required for detection (δ), discrimination, and decision processes (G and β).

4. Fitting Perception–Action Systems

Using the functional data analysis framework (Ramsay & Silverman, 2005), the recorded histories are regarded as collections of functional observations:

$$\begin{aligned} \mathbf{y}_i &= y_i(\mathbf{t}_i) \quad \text{(observed output)}, \\ \mathbf{u}_i &= u_i(\mathbf{t}_i) \quad \text{(input)}. \end{aligned} \tag{10}$$

Each input stream $u_i(\mathbf{t}_i)$ may be decomposed by representing it as a consecutive series of step functions, where the number s_i of step functions vary across individuals. The breaks of these step functions are denoted by $b_{i0}, b_{i1}, b_{i2}, \ldots, b_{is_i}$, where each break pair, $(b_{ij}, b_{i\,j+1})$ for $j = 1, 2, \ldots, s_i - 1$, corresponds to an interval of time during which the stimulus remains constant. By convention, the breaks b_{i0} and b_{is_i} correspond to the first and last observation times for the ith participant.

Using these break points, a partition for the time vector $\mathbf{t}_i$ is constructed:

$$\mathbf{t}_i = \left(\mathbf{t}_{i0}, \mathbf{t}_{i1}, \mathbf{t}_{i2}, \ldots, \mathbf{t}_{is_i}\right)', \tag{11}$$

with the times $\mathbf{t}_{ij}$ contained within $[b_{ij}, b_{i\,j+1})$. Thus, the pitch vector $\mathbf{u}_{ij} = u_i(\mathbf{t}_{ij})$ corresponds to the slider motion $\mathbf{y}_{ij} = y_i(\mathbf{t}_{ij})$. If the sequence of perceived pitches (log-scale) within the input history $\mathbf{u}_i$ is denoted by $p_{i0}, p_{i1}, \ldots, p_{is_i}$, then the jth input step function for participant i may be expressed in terms of these pitch values:

$$u_{ij}(t) = \begin{cases} p_{i\,j-1} & \text{if } t \le b_{ij}, \\ p_{ij} & \text{if } t \in (b_{ij}, b_{i\,j+1}). \end{cases} \tag{12}$$

Figure 3 displays the partitioning scheme described above.

In Section 3, a first-order step response model was proposed to describe the output of a PA system. To relate the pitch tracking data to this setting, the observed output $y_{ij}(t)$ and input $u_{ij}(t)$ histories are translated to the origin:

$$y^0_{ij}(t) = y_{ij}(t) - y_{ij}(b_{ij}), \qquad u^0_{ij}(t) = u_{ij}(t) - p_{ij-1}, \tag{13}$$

for $t \in [b_{ij}, b_{ij+1})$, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, s_i$.

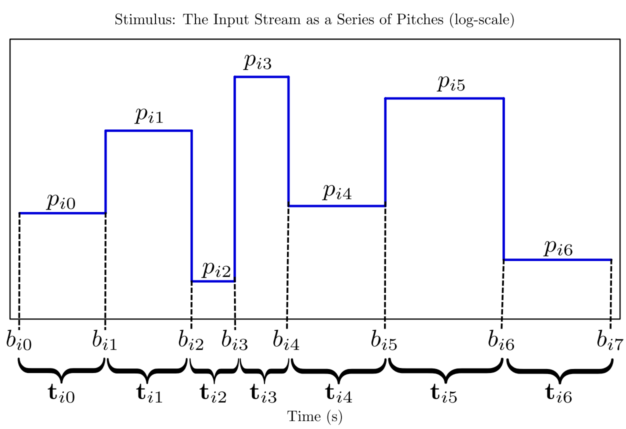


FIGURE 3.
The partitioning of the data according to the known break points $b_{i.}$ and pitches p_{ij}. Each partition of the data is analyzed using a first-order differential equation.

Given a time $t \in \mathbf{t}_{ij}$, the observed slider motion for participant i is modelled as the sum of a systematic and random component:

$$y_{ij}^{0}(t) = x_{ij}(t) + \varepsilon_{ij}(t). \tag{14}$$

For the slider experiment, we let the systematic term $x_{ij}(t)$ be a first-order step response in the form of (7) with structural parameters δ_{ij}, τ_{ij} and G_{ij}:

$$\begin{aligned} \tau_{ij}\mathrm{D}x_{ij}(t) &= -x_{ij}(t) + G_{ij}u_{ij}^{0}(t-\delta_{ij}), \\ x_{ij}(b_{ij}) &= 0. \end{aligned} \tag{15}$$

It is assumed that the random components $\varepsilon_{ij}(t)$ for $t \in \mathbf{t}_{ij}$ are independent and equally distributed random variables with mean zero and covariance matrix $\Sigma_{ij} = \sigma_i^2 \mathrm{I}_{s_i \times s_i}$. Equation (15) is valid for all observed times $t \in \mathbf{t}_{ij}$, and its solution is given as

$$x_{ij}(t) = \begin{cases} 0 & \text{if } t < \delta_{ij} + b_{ij}, \\ G_{ij}(p_{ij} - p_{i\,j-1})\big(1 - \exp\{-\frac{(t-\delta_{ij}-b_{ij})}{\tau_{ij}}\}\big) & \text{if } t \geq \delta_{ij} + b_{ij}. \end{cases} \tag{16}$$

4.1. Estimation by Profiling

The partitioning of the data histories allows the collection of experimental runs for participant i to be treated as a series of consecutive and independent PA systems. The idea is to estimate the reaction times δ_{ij}, the response times τ_{ij} and the steady-state gains G_{ij} separately for each PA system.

One may estimate the structural parameters of our model by minimizing a nonlinear residual sum of squares based on (16) by numerical techniques (Pinheiro & Bates, 1995). However, we wish to present a novel approach that can be applied when dealing with systems of differential equations. For example, we may have a situation where the governing set of differential equations contain nonlinear terms and possibly multiple output and input functions. In such settings, numerical approximation techniques can be time-consuming and complicate the construction of a residual sum of squares.

The generalized profiling technique of Ramsay, Hooker, Campbell and Cao (in press) may be used to estimate the structural parameters of differential equations since it does not rely on finding the solution $x(t)$. Also, the technique is useful for fitting differential equations when the initial conditions are unknown. We demonstrate this procedure within the slider experiment, where the initial conditions of the PA systems are artificially set to zero by (13).

The technique begins with the assumption that every output function $x_{ij}(t)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, s_i$, can be represented by a basis function expansion of the form

$$x_{ij}(t) = \mathbf{c}_{ij}'\boldsymbol{\phi}_{ij}(t) = \boldsymbol{\phi}_{ij}'(t)\mathbf{c}_{ij}, \tag{17}$$

where $\boldsymbol{\phi}_{ij}(t)$ is a vector of B-spline basis functions of order q. To obtain an accurate representation for the solution of (15), the number of basis functions in (17) is allowed to vary and is denoted by K_{ij}. The vector $\mathbf{c}_{ij}$ is a collection of constant coefficients that need to be estimated using the data. For the pitch tracking experiment, B-splines of order $q = 6$ with knots placed every 0.05 seconds were used, allowing us to adequately approximate derivatives of $x(t)$ up to order 4. We shall see later that these parameters provide flexible approximations to the solutions x_{ij}, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, s_i$.

If Φ_{ij} denotes the matrix of basis function values at the observation times $\mathbf{t}_{ij}$, $\Phi_{ij} = \boldsymbol{\phi}_{ij}'(\mathbf{t}_{ij})$, then (17) may be expressed in matrix form, $\mathbf{x}_{ij} = \Phi_{ij}\mathbf{c}_{ij}$. If the structural parameters were known, the coefficient vector $\mathbf{c}_{ij}$ could be estimated directly by minimizing the following residual sum of squares:

$$\mathrm{SSE}\big(\mathbf{y}_{ij}^0|\mathbf{c}_{ij}\big) = \big[\mathbf{y}_{ij}^0 - x_{ij}(\mathbf{t_{ij}})\big]' W_{ij}\big[\mathbf{y}_{ij}^0 - x_{ij}(\mathbf{t_{ij}})\big] = \big[\mathbf{y}_{ij}^0 - \Phi_{ij}\mathbf{c}_{ij}\big]' W_{ij}\big[\mathbf{y}_{ij}^0 - \Phi_{ij}\mathbf{c}_{ij}\big], \quad (18)$$

where W_{ij} is an appropriate weight matrix. The resulting estimated coefficients give an approximation to the response $x_{ij}(t)$ that follows the data closely, but produce rough derivative estimates.

Also, the basis expansion (17) of $x_{ij}(t)$ should follow the first-order step response model with parameters δ_{ij}, τ_{ij} and G_{ij}. Let L_τ be the differential operator defined as

$$L_\tau x(t) = \tau \mathrm{D}x(t) + x(t), \quad (19)$$

so that $x_{ij}(t)$ satisfies

$$L_{\tau_{ij}} x_{ij}(t) = G_{ij} u^0(t - \delta_{ij}). \quad (20)$$

Assuming the structural parameters of the model are known, the extent to which our basis expansion satisfies (20) is assessed by the roughness penalty or fidelity:

$$\mathrm{PEN}(\mathbf{c}_{ij}|\boldsymbol{\theta}_{ij}) = \int_{b_{ij}}^{b_{i\,j+1}} \big[L_{\tau_{ij}} \mathbf{c}_{ij}' \boldsymbol{\phi}_{ij}(s) - G_{ij} u^0(s - \delta_{ij})\big]^2 \,\mathrm{d}s. \quad (21)$$

Expression (21) may be expanded further by defining the following objects:

- $R(\tau_{ij})$ is the order K_{ij} symmetric matrix

$$R(\tau_{ij}) = \int_{b_{ij}}^{b_{i\,j+1}} \big[L_{\tau_{ij}} \boldsymbol{\phi}_{ij}(s)\big]\big[L_{\tau_{ij}} \boldsymbol{\phi}_{ij}(s)\big]' \,\mathrm{d}s \quad (22)$$

- $\mathbf{s}(\boldsymbol{\theta}_{ij})$ is the vector

$$\mathbf{s}(\boldsymbol{\theta}_{ij}) = G_{ij} \int_{b_{ij}}^{b_{i\,j+1}} \big[L_{\tau_{ij}} \boldsymbol{\phi}_{ij}(s)\big] u_{ij}^0(s - \delta_{ij}) \,\mathrm{d}s \quad (23)$$

- $A(\boldsymbol{\theta}_{ij})$ is the scalar

$$A(\boldsymbol{\theta}_{ij}) = G_{ij}^2 \int_{b_{ij}}^{b_{i\,j+1}} \big[u_{ij}^0(s - \delta_{ij})\big]^2 \,\mathrm{d}s. \quad (24)$$

The fidelity $\mathrm{PEN}(\mathbf{c}_{ij}|\boldsymbol{\theta}_{ij})$ can be reexpressed as

$$\mathrm{PEN}(\mathbf{c}_{ij}|\boldsymbol{\theta}_{ij}) = \mathbf{c}_{ij}' R(\tau_{ij}) \mathbf{c}_{ij} - 2\mathbf{c}_{ij}' \mathbf{s}(\boldsymbol{\theta}_{ij}) + A(\boldsymbol{\theta}_{ij}), \quad (25)$$

and if each observed PA system follows a first-order step response, then the basis expansion of $x_{ij}(t)$ should yield a relatively small fidelity value.

In functional data analysis, the fidelity is often used to create a smooth version of the basis expansion of $x_{ij}(t)$, a process known as regularization. This is accomplished by combining the fidelity, $\mathrm{PEN}(\mathbf{c}_{ij}|\boldsymbol{\theta}_{ij})$, with the usual residual sum of squares, $\mathrm{SSE}(\mathbf{y}_{ij}|\mathbf{c}_{ij})$, and then minimizing the resulting penalized form of the residual sum of squares:

$$\begin{aligned}\mathrm{PENSSE}_\lambda\big(\mathbf{y}_{ij}^0|\mathbf{c}_{ij}, \boldsymbol{\theta}_{ij}\big) &= \mathrm{SSE}\big(\mathbf{y}_{ij}^0|\mathbf{c}_{ij}\big) + \lambda\,\mathrm{PEN}(\mathbf{c}_{ij}|\boldsymbol{\theta}_{ij}) \\ &= \big[\mathbf{y}_{ij}^0 - \Phi_{ij}\mathbf{c}_{ij}\big]' W_{ij}\big[\mathbf{y}_{ij}^0 - \Phi_{ij}\mathbf{c}_{ij}\big] + \lambda \mathbf{c}_{ij}' R(\tau_{ij}) \mathbf{c}_{ij} \\ &\quad - 2\lambda \mathbf{c}_{ij}' \mathbf{s}(\boldsymbol{\theta}_{ij}) + \lambda A(\boldsymbol{\theta}_{ij}). \end{aligned} \quad (26)$$

The smoothing parameter λ in (26) controls how the response $x_{ij}(t)$ satisfies the system as defined in (21).

The least-squares estimate of the coefficient vector is expressed as

$$\widehat{\mathbf{c}}_{ij}(\boldsymbol{\theta}_{ij}|\mathbf{y}_{ij},\lambda) = \left[\Phi'_{ij}W_{ij}\Phi_{ij} + \lambda R(\tau_{ij})\right]^{-1}\left[\Phi'_{ij}W_{ij}\mathbf{y}^0_{ij} + \lambda s(\boldsymbol{\theta}_{ij})\right], \tag{27}$$

and the fit to the data $\mathbf{y}_{ij}$ is

$$\widehat{\mathbf{x}}_{ij} = \Phi_{ij}\left[\Phi'_{ij}W_{ij}\Phi_{ij} + \lambda R(\tau_{ij})\right]^{-1}\left[\Phi'_{ij}W_{ij}\mathbf{y}^0_{ij} + \lambda s(\boldsymbol{\theta}_{ij})\right]. \tag{28}$$

When $\lambda = 0$, this fit gives the usual (unregularized) least-squares estimate of $\mathbf{c}_{ij}$ and the resulting expansion of $x_{ij}(t)$ is treated as an interpolator to the response history. As $\lambda \to \infty$, more weight is put on the fidelity and the regression curve is restricted to the true solution of the differential equation (16).

What is of importance is the estimation of the structural parameters of the differential equations and not the coefficient vector $\mathbf{c}_{ij}$ of the basis expansions. The latter are treated as nuisance parameters, and are removed from the estimation of δ_{ij}, τ_{ij} and G_{ij} by employing unrestricted and penalized residual sum of squares in a recursive two-step procedure known as profiling.

The unrestricted residual sum of squares is

$$\mathrm{SSE}(\mathbf{y}_{ij}|\mathbf{x}_{ij}) = \left[\mathbf{y}^0_{ij} - x_{ij}(\mathbf{t_{ij}})\right]' W_{ij}\left[\mathbf{y}^0_{ij} - x_{ij}(\mathbf{t_{ij}})\right], \tag{29}$$

and since we require that the basis function expansion of $x_{ij}(t)$ be a solution to the differential equation (15), this is necessarily a function of the coefficient vector $\mathbf{c}_{ij}$ and the structural parameter vector $\boldsymbol{\theta}_{ij}$. In principle, joint estimates of the unknown coefficient vector and structural parameters may be obtained by minimizing this objective function. However, due to the very high dimension of this parameter space, joint estimation is often cumbersome. For example, a typical expansion of x_{ij} for the pitch tracking experiment requires on the order of 100 coefficients.

Profiling requires that for any change in $\boldsymbol{\theta}_{ij}$, $\mathrm{SSE}(\mathbf{y}_{ij}|\mathbf{x}_{ij})$ is minimized with respect to the coefficient vector $\mathbf{c}_{ij}$. This defines a one-to-one mapping from $\boldsymbol{\theta}_{ij}$ to $\mathbf{c}_{ij}$ and reduces the dimension of the estimation problem by treating the nuisance parameters as functions of the structural parameters and the data vector $\mathbf{y}^0_{ij}$:

$$\mathbf{c}_{ij} = \mathbf{c}_{ij}\left(\boldsymbol{\theta}_{ij}|\mathbf{y}^0_{ij}\right). \tag{30}$$

Substituting (30) into (29) we obtain the profiled version of the residual sum of squares

$$\mathrm{SSE}_{\boldsymbol{\theta}_{ij}} = \mathrm{SSE}\left(\mathbf{y}^0_{ij}|\mathbf{x}_{ij} = \mathbf{c}_{ij}(\boldsymbol{\theta}_{ij})'\boldsymbol{\phi}_{ij}(\mathbf{t}_{ij})\right). \tag{31}$$

In elementary cases, the closed-form expression of $\mathbf{c}_{ij}(\boldsymbol{\theta}_{ij}|\mathbf{y}^0_{ij})$ is readily available as is the case for the first-order step response system. For more complex systems where the closed form is often elusive, one can use an inner optimization of PENSSE_λ to define the relationship between the coefficient vector $\mathbf{c}_{ij}$ and the structural parameters $\boldsymbol{\theta}_{ij}$.

The overall optimization procedure is obtained by combining the two optimizations:

$$\widehat{\boldsymbol{\theta}}_{ij} = \arg\min_{\boldsymbol{\theta}_{ij}} \mathrm{SSE}_{\boldsymbol{\theta}_{ij}} \tag{32}$$

and

$$\mathbf{c}_{ij}(\boldsymbol{\theta}_{ij}) = \arg\min_{c_{ij}} \mathrm{PENSSE}_\lambda. \tag{33}$$

The inner optimization criterion (33) provides a coefficient vector $\mathbf{c}_{ij}(\widehat{\boldsymbol{\theta}}_{ij})$ that ensures a relatively small fidelity as controlled by the smoothing parameter λ. This is useful in cases where a model is known to be wrong but is still desired because of its interpretative and descriptive properties. This is the case with our use of the first-order step response in the pitch tracking experiment were obvious factors, such as friction and second-order harmonics, have been ignored.

A numerical optimization procedure may be used to calculate the estimates $\widehat{\boldsymbol{\theta}}_{ij}$ that minimize the outer optimization criterion (32). The calculation of a gradient, which may be used in the optimization steps given above, is described in the Appendix.

The generalized profiling procedure above assumes that a value for the bandwidth parameter λ was chosen. Thus far, there is no known general rule for bandwidth selection that would apply to most situations for fitting differential equations. In cases such as the slider experiment, where the model specification offers a partial explanation to the slider dynamics, (Ramsay et al., in press) suggest that the value of λ can be chosen by visual inspection to fit the data. In order to facilitate the computation, a bandwidth value of $\lambda = 100$ was chosen. This was adequate to represent the majority of the first-order step responses with basis expansions of B-splines. This was assessed by inspecting all observed responses versus the solutions of the first-order step responses using the parameter estimates obtained from the profiling method.

5. Calibrating the Dynamics

Figure 4 shows two fitted responses for PA systems for participants 1 and 16. On average, approximately 50 estimates for each structural parameter were obtained per individual and these are presented in strip-chart format in Figure 5. The mean (s.e.) reaction time δ was 0.75 (0.009), the mean (s.e.) response time τ was 0.39 (0.015), and the mean (s.e.) gain G was 2.91 (0.048). However, an examination of Figure 5 reveals that the estimates of the structural parameters exhibit two sources of variation: between and within participants. To remedy this, a calibration of the parameters is needed to obtain meaningful summary statistics of the structural parameters.

5.1. Calibrating with Multilevel Models

Fitts' law (Fitts, 1954) describes the speed–accuracy trade-off in human movement and states that a positive linear relationship exists between the response time τ and the logarithm of the distance of movement. Since participants in the tracking experiment map sonic stimuli to the slider range, it is reasonable to assume that their tracking dynamics are affected by the magnitude of the perceived pitch changes, $\Delta p_{ij} = p_{ij} - p_{ij-1}$. A graphical examination of our estimates suggests that the logarithms of the structural parameters are linearly dependent on the magnitude of change in perceived pitch change. Thus, a calibration of the parameters must be conducted in order to take this dependency into account.

By ignoring participant differences, we run the risk of misrepresenting the true experiment variation. This is apparent if we consider the precalibration standardized residuals obtained by fitting linear regression models, one for each structural parameter. In Figure 6, box-plots of these residuals are grouped by participant, and these show that individual differences tend to shift averages away from the origin and that the variances within subjects differ considerably. There are many reasons why the estimates exhibit this behaviour, and part of it is due to the fact that each participant employs their own range of the slider device. Also, uncontrollable experimental conditions may contribute to within-participant variation. A partial remedy for these individual differences is to propose a model for heteroscedasticity.

A true calibration consists of treating the estimates as hierarchical data, which assumes that the perception action systems are nested within participants of the experiment. Two-stage linear

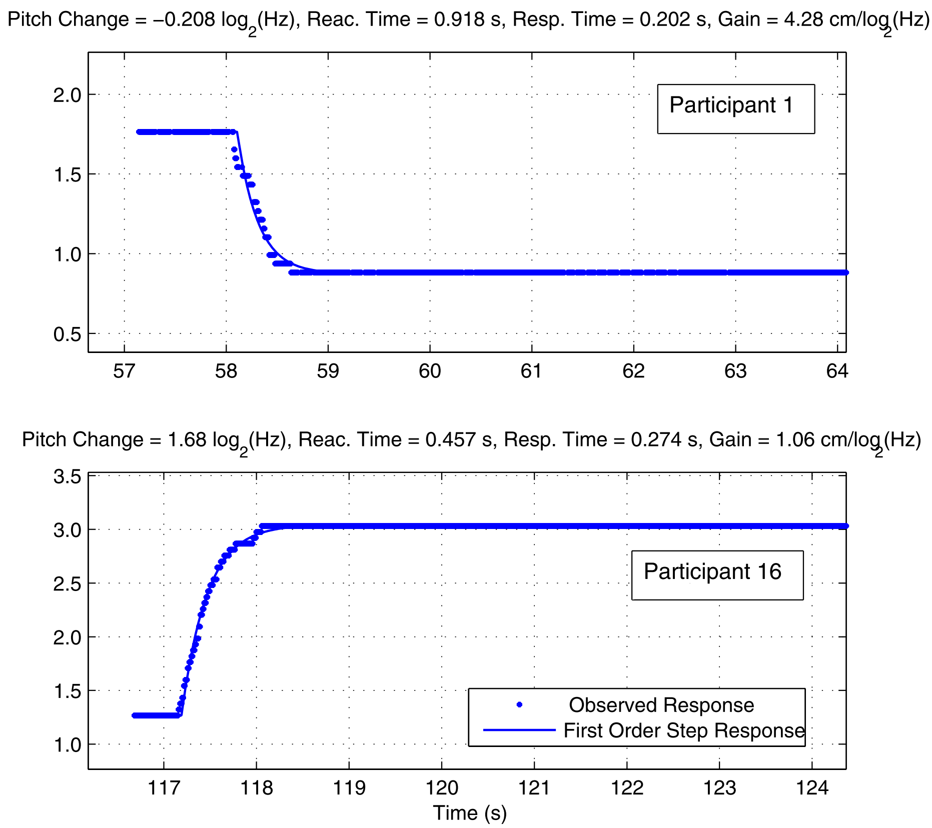


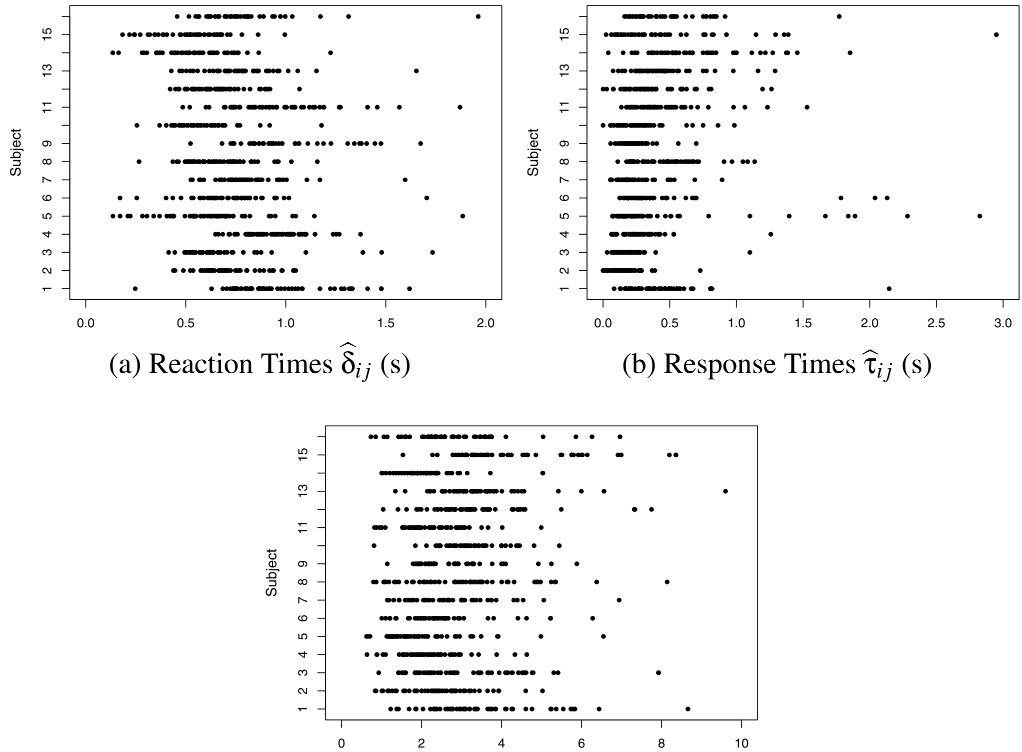
FIGURE 4.
Two first-order step response fits for PA systems corresponding to participants 1 (*top*) and 16 (*bottom*). The smoothing parameter $\lambda = 100$ was used in PENSSE$_\lambda$.

mixed-effects models were employed to identify individual and population characteristics, which results in a calibration of the structural parameters.

A centring of the regressors towards the origin was used to reduce correlations between the intercept and slope estimates. In our situation, a value of 1.08 was adequate to centre the perceived pitch changes $|\Delta p_{ij}|$. The calibration models had the following form:

$$z_{ij} = \big(m^0 + \mu_i^0\big) + \big(m^1 + \mu_i^1\big)\big(|\Delta p_{ij}| - 1.08\big) + \varepsilon_{ij},$$
$$\varepsilon_{ij} \sim \mathcal{N}\big(0, \sigma^2 \nu_i^2\big), \quad \boldsymbol{\mu}_i = \begin{bmatrix} \mu_i^0 \\ \mu_i^1 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{\psi}\right),$$
$$\boldsymbol{\psi} = \begin{bmatrix} \psi_0^2 & \psi_{01} \\ \psi_{10} & \psi_1^2 \end{bmatrix}, \tag{34}$$

where z_{ij} was one of $\log(\delta_{ij})$, $\log(\tau_{ij})$ or $\log(G_{ij})$. The coefficients μ^0 and μ^1 represent the population intercept and slope, and the random effects capture the influence of the ith individual on their PA systems, with μ_i^0 and μ_i^1 representing the participant's "natural" slider location and pitch change effect, respectively. The random-effects vectors $\boldsymbol{\mu}_i$ are assumed independent.



(a) Reaction Times $\widehat{\delta}_{ij}$ (s) (b) Response Times $\widehat{\tau}_{ij}$ (s)

(c) Gains $\widehat{G}_{ij}$ (cm/log_2Hz)

FIGURE 5.
Estimates of the structural parameters, grouped by participant.

The errors ε_{ij} are assumed independent between individuals and an autocorrelation analysis revealed little evidence of serial correlation in the estimates within individuals. The heteroscedasticity parameters ν_i, $i = 1, 2, \ldots, 16$, are used to capture individual slider ranges. A restriction $\nu_1 = 1$ is required to ensure identifiability, and under this constraint the variance parameter ν_i is interpreted as the ratio between the standard deviations of the ith participant to the first.

5.2. *Results and Discussion*

A calibration of the estimates was performed using the *nlme* package (version 3.1.68.1) of the *R* statistical computation and graphics system (version 2.1.1). The final models were chosen using a forward regression approach with repeated likelihood-ratio tests for model selection. In depth information on the theory and estimation of mixed-effects models can be found in Pinheiro and Bates (2000).

The mixed-effects estimates and 95% confidence intervals for the calibration models for reaction time, response time and gain are given in Tables 1, 2 and 3, which include within-subject and between-subject standard errors. We discovered that the random effects variance–covariance matrix could be reduced to the case where $\psi_{01} = \psi_{10} = 0$ for all calibrations, suggesting that no correlations between random effects exist in the calibration models. For reaction time, there was no evidence to suggest that random effects were present for pitch change and, in this case, we set $\psi_1^2 = 0$.

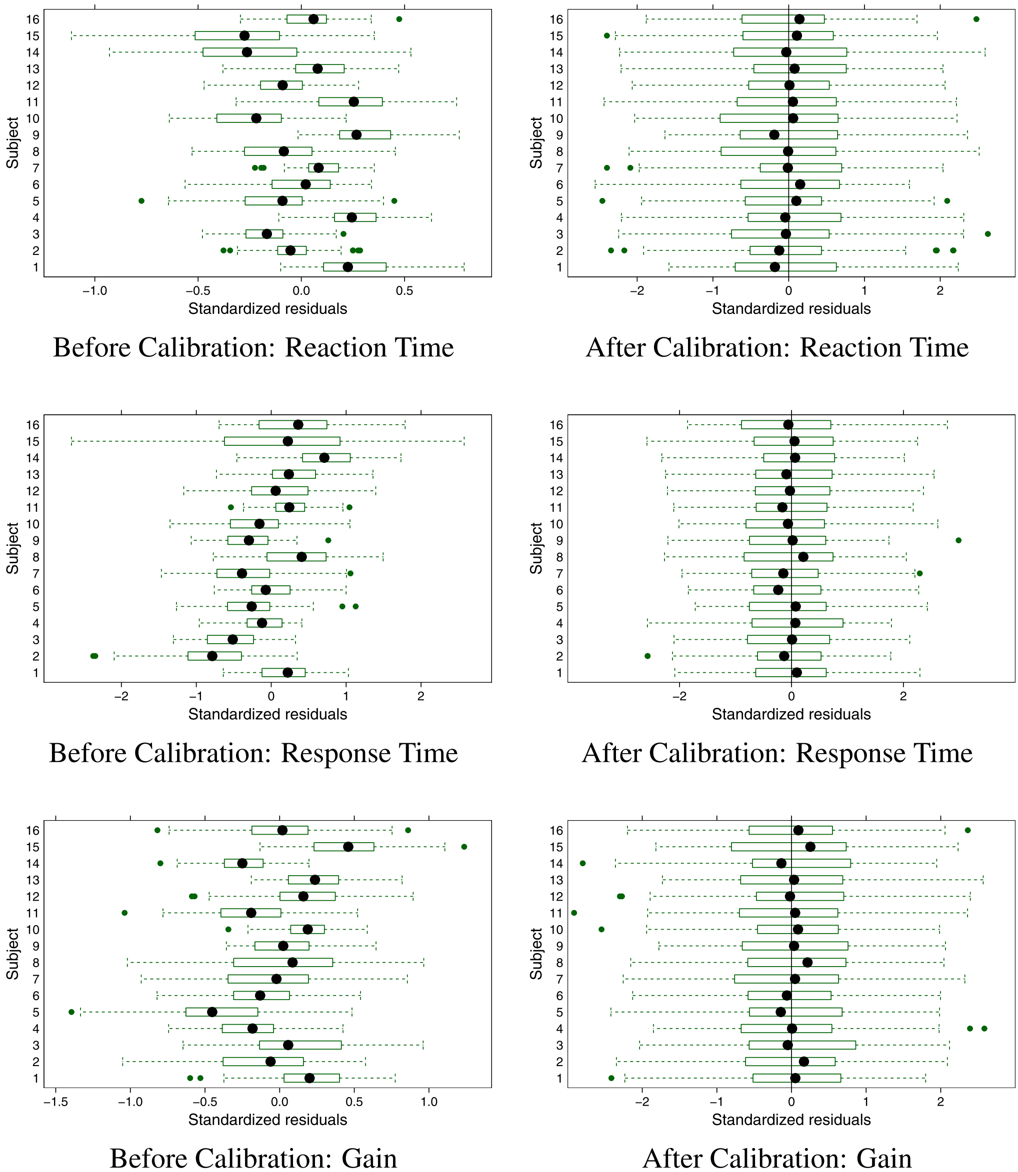


FIGURE 6.
Box plots of standardized residuals, grouped by participant.

TABLE 1.
Reaction time calibration: Mixed-effects estimates.

	Value	Std. error	DF	t-value
m^0	−0.345	0.049	764	−7.1
m^1	−0.053	0.012	764	−4.3

Random effects: 95% confidence intervals

	Lower	Upper		Lower	Est.	Upper
μ^0	−0.440	−0.249	$\widehat{\sigma}$	0.20	0.23	0.27
μ^1	−0.077	−0.029	$\widehat{\psi}_0$	0.13	0.19	0.27

TABLE 2.
Response time calibration: Mixed-effects estimates.

		Value	Std. error	DF	t-value	
	m^0	-1.22	0.091	771	-13.4	
	m^1	0.30	0.061	771	4.8	
Random effects: 95% confidence intervals						
	Lower	Upper		Lower	Est.	Upper
μ^0	-1.40	-1.04	$\widehat{\sigma}$	0.48	0.61	0.78
μ^1	0.18	0.42	$\widehat{\psi}_0$	0.29	0.38	0.49
			$\widehat{\psi}_1$	0.11	0.21	0.38

TABLE 3.
Gain calibration: Mixed-effects estimates.

		Value	Std. error	DF	t-value	
	m^0	0.98	0.053	775	18.4	
	m^1	-0.19	0.037	775	-5.2	
Random effects: 95% confidence intervals						
	Lower	Upper		Lower	Est.	Upper
μ^0	0.87	1.08	$\widehat{\sigma}$	0.26	0.33	0.40
μ^1	-0.26	-0.12	$\widehat{\psi}_0$	0.144	0.21	0.30
			$\widehat{\psi}_1$	0.072	0.12	0.21

The between-group models for the reaction time, response time and gain were estimated as

$$\overline{\log(\delta)} = -0.345 - 0.053\big(|\Delta p| - 1.08\big) = -0.288 - 0.053|\Delta p|, \tag{35}$$

$$\overline{\log(\tau)} = -1.22 + 0.30\big(|\Delta p| - 1.08\big) = -1.54 + 0.30|\Delta p|, \tag{36}$$

$$\overline{\log(G)} = 0.98 - 0.19\big(|\Delta p| - 1.08\big) = 1.19 - 0.19|\Delta p|, \tag{37}$$

suggesting that the structural parameters of a PA system in our pitch-tracking experiment are dependent on the change in input. The calibration analysis found that reaction time δ and gain G decrease slightly as pitch change increases in absolute value. In contrast, the response time τ increases as $|\Delta p|$ gets larger.

To examine the calibrated fits for δ, τ and G, standardized model residuals were calculated by subtracting the fitted models and estimated random effects vectors, $\boldsymbol{\mu}_i$, from the estimates of the structural parameters δ_i, τ_i and G_i, and dividing by the within-group standard deviations. Box-plots of these residuals are grouped and displayed by participant in the right-hand panels of Figure 5. Comparing these with the precalibration standardized residuals, we see that they are now centred around the origin and that their ranges overlap, an indication that calibration was achieved.

6. Summary

By incorporating derivatives to account for the dynamic nature of the data, we were able to extract information from the pitch-tracking experiment in a meaningful way. This was accomplished by considering the first-order step response as a model, which in turn allowed for a decomposition of individual responses.

At the very least, three components are required to describe continuous responses to stimuli: reaction time, δ, response time, τ, and gain, G. For a given pitch change Δp, we expect the slider position with initial position x^0 to approach a new steady-state position $x^* = x^0 + G\Delta p$ in approximately $\delta + 4\tau$ time units, after a latency period of δ time units.

Using the generalized profiling method, estimates of the components δ, τ, and G were obtained for all participants in the experiment. Under the assumption that the structural parameters are constant, the first-order equation is sufficient to accurately describe the system with any specified forcing function. Thus, the use of step functions in this case was ideal for the estimation of the structural parameters without having to account for different forcing functions.

A calibration of the structural parameters was necessary to normalize responses across participants and to extract population summaries of the reaction time, δ, response time, τ, and gain, G. Calibration also permitted us to explore the relationship between pitch change, Δp, and the slider dynamics. Our findings suggest that the following is a more appropriate model for pitch tracking:

$$\tau(\Delta p)\mathrm{D}x(t) = -x(t) + G(\Delta p)u\bigl(t - \delta(\Delta p)\bigr), \tag{38}$$

such that reaction time, response time and gain are

$$\begin{aligned}\delta(\Delta p) &= \exp\bigl(a^0 + a^1|\Delta p|\bigr),\\ \tau(\Delta p) &= \exp\bigl(b^0 + b^1|\Delta p|\bigr),\\ G(\Delta p) &= \exp\bigl(c^0 + c^1|\Delta p|\bigr),\end{aligned}$$

for constants a^0, a^1, b^0, b^1 , c^0, c^1.

The modified model in (38) gives a better understanding of the human dynamics in pitch tracking, but it is far from complete. This model does not include any terms for damping caused by friction, a natural forcing function that tends to oppose the slider motion. For the pitch tracking experiment, we assumed that the frictional forces were negligible and omitted them from the differential equations. Also, other higher-order terms, such as $\mathrm{D}^2x(t)$, may be necessary to properly explain any damping or oscillations in the motion of the slider.

With calibration, we've accounted for the variation in the responses across participants and stimuli; however, under this reality, changes in the forcing functions lead to different dynamics and, consequently, it is difficult to go back to the first-order differential equation as a general model for pitch tracking. The next step is to rework the PA systems to accommodate for the nonlinear dynamics that were detected in the calibration stage. In addition, we have the machinery to deal with higher-order or nonlinear dynamics which does not depend on closed-form solutions of the differential equations.

As a bonus, calibration allows for the estimation of heteroscedasticity parameters, ν_i, and of the random effects vectors, $\boldsymbol{\mu}_i$, which may be regarded as individual descriptors of performance. These can be used to assess tracking performance, compare individuals and to normalize continuous judgements in experiments such as Vines, et al. (2006), thus reducing residual variation. Individual subject profiles can also be used to study performance styles in the event that two or more different subject types exist in the population (e.g. absolute pitch possessors vs. nonpossessors).

Dynamical systems present new and exciting challenges to traditional statistical thinking about data. In conjunction with estimation techniques, model building and model checking procedures for systems of differential equations are required to expand and improve current models for human cognition.

Appendix

Methods

Participants The 16 participants of the pitch-tracking experiment were recruited from the McGill University community. This group was composed of 13 women and 3 men, between the ages of 21 and 29 years (mean age 24.3 years, s.e. 2.7). All participants were right-handed with 13 out of 16 considering themselves musical. Each participant received 20 dollars CDN for their participation.

Materials *Max/MSP*, a computing environment for audio, was used to generate the sonic stimuli and to record responses. The software ran on an *Apple Powerbook G4* connected to the slider device, a *Peavy* 1600*X* MIDI controller, via a MIDI-to-USB converter. One slider on the controller was active and was used as the input device. To limit any external sounds, participants were asked to listen to the stimuli through a pair of *AKG K*240 headphones. A sound pressure level calibration of the audio chain was performed with a sound level metre and headphone coupler. To account for perceived loudness, the stimuli were adjusted according to equal loudness contours.

Due to constraints in the *MAX/SMP* sampling mechanism, responses were sampled and recorded at 172.266 Hz, roughly every 0.0058 seconds, to minimize quantization error, noise introduced in an analogue to digital conversion.

Procedure The experiment was composed of two phases: the glissando phase, where participants learned to use the slider device, and the melodies phase, where participants were asked to follow a sequence of constant pitches.

The glissando phase consisted of three blocks, each with a single auditory stimulus consisting of a sine tone that sweeps from D3 (146.832) to D6 (1174.66 Hz) on a logarithmic scale of base 2. The sweep times for the blocks were 6, 3 and 1.5 seconds, respectively. Participants were informed that the range of the slider represented the range of pitches, with the lowest and highest pitches corresponding to the bottom and top of the slider, respectively. The blocks were presented in order and participants were instructed to use the slider to follow the sweeps. Before each block, the slider positing was set to the bottom position and participants were informed on the length of the sweep.

The pitch-tracking stage was composed of three blocks, each with a two-minute stream of stimuli. The stimuli were steady-state sine tones generated at random and separated by 20 millisecond linear cross-fades. The length of duration for each stimulus was between 2 and 10 seconds and was sampled from a rectangular distribution. The frequencies of each stimulus were chosen at random on a log-linear scale of base 2 in the three-octave intervals between D3 and D6. One of the three blocks remained fixed for all participants, but was presented at random to minimize any possible order effects. This fixed stimulus is presented in Figure 1.

Each participant was instructed, upon presentation of the stimuli, to follow the sine tones, in the same range as the glissandi, by adjusting a slider on the *Peavy* 1600*X* MIDI controller. Before each block, the slider position of the controller was set to the bottom position. After the experiment, all participants completed a musical questionnaire and the Edinburgh Handedness Inventory (Oldfield, 1971).

Gradient Calculation

When employing the Gauss–Newton method, a gradient $D_{\boldsymbol{\theta}_{ij}} G$ of the outer objective function (32) is required. The gradient of G with respect to $\boldsymbol{\theta}_{ij}$ is

$$D_{\boldsymbol{\theta}_{ij}} \mathrm{SSE}_{\boldsymbol{\theta}_{ij}} = \frac{\partial\, \mathrm{SSE}_{\boldsymbol{\theta}_{ij}}}{\partial \boldsymbol{\theta}} + \frac{\partial\, \mathrm{SSE}_{\boldsymbol{\theta}_{ij}}}{\partial \mathbf{c}_{ij}} \frac{\mathrm{d}\mathbf{c}_{ij}}{\mathrm{d}\boldsymbol{\theta}_{ij}}. \tag{39}$$

In the case of the first-order step response, the gradient in (39) simplifies by noting that the first term, $\partial\,\mathrm{SSE}_{\boldsymbol{\theta}_{ij}}/\partial\boldsymbol{\theta}$, is zero and that a closed-form expression of $\mathrm{d}\mathbf{c}_{ij}/\mathrm{d}\boldsymbol{\theta}_{ij}$ exists. For more complex systems, the gradient $D_{\boldsymbol{\theta}_{ij}}\,\mathrm{SSE}_{\boldsymbol{\theta}_{ij}}$ may be calculated with the use of the implicit function theorem. To keep the notation compact, the subscripts from $\boldsymbol{\theta}_{ij}$ and $\mathbf{c}_{ij}$ and any references to $\mathbf{y}_{ij}$ are temporarily omitted. Let $g = D_{\boldsymbol{\theta}}\,\mathrm{SSE}_{\boldsymbol{\theta}}$ and $h = D_{\mathbf{c}}\,\mathrm{PENSSE}_{\lambda}$ denote the gradients of the outer and inner objective functions:

$$g(\boldsymbol{\theta}) = \frac{\partial\,\mathrm{SSE}_{\boldsymbol{\theta}}}{\partial\boldsymbol{\theta}} + \frac{\partial\,\mathrm{SSE}_{\boldsymbol{\theta}}}{\partial\mathbf{c}}\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}\boldsymbol{\theta}} \tag{40}$$

and

$$h(\mathbf{c}|\boldsymbol{\theta}) = \frac{\partial\,\mathrm{PENSSE}_{\lambda}}{\partial\mathbf{c}}. \tag{41}$$

Then, for a fixed value of $\boldsymbol{\theta}$, the optimal value of $\mathbf{c}$ must satisfy

$$h(\mathbf{c}|\boldsymbol{\theta}) = 0. \tag{42}$$

Taking the gradient of (42) with respect to $\boldsymbol{\theta}$ gives

$$D_{\boldsymbol{\theta}}h(\mathbf{c}|\boldsymbol{\theta}) = \frac{\partial h}{\partial\boldsymbol{\theta}} + \frac{\partial h}{\partial\mathbf{c}}\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}\boldsymbol{\theta}} = 0, \tag{43}$$

and this implies that

$$\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}\boldsymbol{\theta}} = -\left(\frac{\partial h}{\partial\mathbf{c}}\right)^{-1}\frac{\partial h}{\partial\boldsymbol{\theta}} = -\left(\frac{\partial^2\,\mathrm{PENSSE}_{\lambda}}{\partial\mathbf{c}^2}\right)^{-1}\frac{\partial^2\,\mathrm{PENSSE}_{\lambda}}{\partial\mathbf{c}\partial\boldsymbol{\theta}}. \tag{44}$$

Thus, one obtains a useful version for the gradient of $\mathrm{SSE}_{\boldsymbol{\theta}}$ by substituting (44) in (40):

$$g(\boldsymbol{\theta}) = \frac{\partial\,\mathrm{SSE}_{\boldsymbol{\theta}}}{\partial\boldsymbol{\theta}} - \frac{\partial\,\mathrm{SSE}_{\boldsymbol{\theta}}}{\partial\mathbf{c}}\left(\frac{\partial^2\,\mathrm{PENSSE}_{\lambda}}{\partial\mathbf{c}^2}\right)^{-1}\frac{\partial^2\,\mathrm{PENSSE}_{\lambda}}{\partial\mathbf{c}\partial\boldsymbol{\theta}}. \tag{45}$$

References

Boker, S.M., & Wenger, M.J. (2007). *Data analytic techniques for dynamical systems*. Mahwah: Lawrence Erlbaum.

Chow, S.-M., Ferrer, E., & Nesselroade, J.R. (in press). An unscented Kalman filter approach to the estimation of nonlinear dynamical systems models. *Multivariate Behavioral Research*.

Fitts, P.M. (1954). The information capacity of the human motor system in controlling the amplitude of movement. *Journal of Experimental Psychology*, *47*, 381–391.

Grodins, F.S. (1963). *Control theory and biological systems*. New York: Columbia University Press.

Jagacinski, R.J., & Flach, J.M. (2003a). *Control theory for humans*. Mahwah: Lawrence Erlbaum.

Jagacinski, R.J., & Flach, J.M. (2003b). *Control theory for humans: Quantitative approaches to modeling performance*. Mahwah: Lawrence Erlbaum.

Madsen, C.K., & Fredrickson, W.E. (1993). The experience of musical tension: A replication of Nielsen's research using the continuous response digital interface. *Journal of Music Therapy*, *30*, 46–63.

Marken, R.S. (2001). Controlled variables Psychology as the center fielder views it. *The American Journal of Psychology*, *114*, 259–281.

Newell, K.W., & Molenaar, P.C.M. (1998). *Applications of nonlinear dynamics to developmental process modeling*. Mahwah: Lawrence Erlbaum.

Oldfield, R.C. (1971). The assessment and analysis of handedness: The Edinburgh inventory. *Neuropsychologia*, *9*, 97–113.

Pinheiro, J.C., & Bates, D.M. (1995). *Model building for nonlinear mixed-effects models*. (Technical Report, No. 91). Madison: Department of Biostatistics, University of Wisconsin.

Pinheiro, J.C., & Bates, D.M. (2000). *Mixed-effects models in S and S-PLUS*. New York: Springer.

Powers, W. (1973). *Behavior: The control of perception*. Chicago: Aldine Publishing.

Powers, W. (1990). Control theory and statistical generalizations. *American Behavioral Scientist*, *34*, 24–31.

Ramsay, J.O., & Silverman, B.W. (2005). *Functional data analysis* (2nd ed.). New York: Springer.
Ramsay, J.O., Hooker, G., Campbell, D., & Cao, J. (in press). Parameter estimation for differential equations: A generalized smoothing approach, with discussion. *Journal of the Royal Statistical Society, Series B*.
Runkel, P.L. (1990). Research method for control theory. *American Behavioral Scientist*, *34*, 14–23.
Searle, L.V., & Taylor, F.V. (1948). Studies of tracking behaviour. I. Rate and time characteristics of simple corrective movements. *Journal of Experimental Psychology*, *38*, 615–631.
Stevens, S.S., & Volkmann, J. (1940). The relation of pitch to frequency: A revised scale. *American Journal of Psychology*, *53*, 329–353.
Taylor, F.V., & Birmingham, H.P. (1948). Studies of tracking behavior. II. The acceleration pattern of quick manual corrective responses. *Journal of Experimental Psychology*, *38*, 783–795.
Tustin, A. (1952). Feedback. In *Mathematical thinking in behavioral sciences: Readings from Scientific American* (pp. 66–73). San Francisco: Freeman (with introductions by D.M. Messick (1968)).
Vines, B.W., Nuzzo, R.L., & Levitin, D.J. (2005). Analyzing temporal dynamics in music: Differential calculus, physics, and functional data analysis techniques. *Music Perception*.
Vines, B.W., Krumhansl, C.L., Wanderley, M.M., & Levitin, D.J. (2006). Cross-modal interactions in the perception of musical performance. *Cognition*, *101*, 80–113.
Walls, A.T., & Schafer, J.L. (2006). *Models for intensive longitudinal data*. New York: Oxford University Press.
Wiener, N. (1948). *Cybernetics*. New York: Wiley.

Manuscript received 9 MAY 2006
Final version received 5 MAR 2007
Published Online Date: 22 SEP 2007